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Square Hamiltonian cycles in graphs with maximal 4-cliques

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Abstract

The square of a graph is obtained by adding additional edges joining all pairs of vertices with distance two in the original graph. Pósa conjectured that if G is a simple graph on n vertices with minimum degree $2n/3$, then G contains the square of a hamiltonian cycle. We show that Pósa's conjecture holds for graphs that in addition contain a maximal 4-clique.

0. Introduction

In this article we shall only consider simple graphs. The minimum degree of a graph G is denoted by $\delta(G)$. A k-chord of a cycle or path C is an edge joining two vertices of distance k in C. The kth power of C is the graph obtained by joining every pair of vertices with distance at most k. The second power of C is called the square of C. A square cycle (path) is the square of a cycle (path). A hamiltonian square cycle (path) is the square of a hamiltonian cycle (path). Let G be any graph on n vertices. Dirac's famous theorem [1] asserts that if $\delta(G) \geqslant n/2$, then G is hamiltonian. In 1963 Pósa (see [2]) conjectured that if $\delta(G) \geqslant 2n/3$, then G contains a hamiltonian square cycle. The complete tripartite graph $K_{t,t,t-1}$ on $3t-1$ vertices has minimum degree $2t-1=2(3t-1)/3-1/3$, but does not contain a hamiltonian square cycle. So, if true, the conjecture is best possible. In 1973 Seymour [8] made the more general conjecture that if $\delta(G) \geqslant kn/(k+1)$, then G contains the kth power of a hamiltonian cycle. Until recently, essentially no progress was made on Pósa's conjecture. Häggkvist showed, but did not publish, that the conjecture was true if 2/3 was weakened to 3/4. Then Fan and Häggkvist [3] showed that 5/7 was enough. Then Fan and Kierstead [4] proved the following asymptotic version of the conjecture.

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Theorem 0.1. *For every $\varepsilon > 0$ there exists an integer m, depending only on ε, such that if G is a graph on n vertices with $\delta(G) \geqslant (\frac{2}{3} + \varepsilon)n + m$, then G contains a hamiltonian square cycle.*

In [5] Fan and Kierstead proved the next theorem that shows that a slightly weaker hypothesis on the minimum degree is enough to insure a hamiltonian square path. This weaker hypothesis will be crucial for the proof of the main result of this paper.

Theorem 0.2. *Let G be a graph on n vertices with $\delta(G) \geqslant (2n-1)/3$. Then G contains a hamiltonian square path.*

We shall also need the next theorem proved by Fan and Kierstead in [6].

Theorem 0.3. *Let G be a graph on n vertices with $\delta(G) \geqslant 2n/3$. If G contains a square cycle of length greater than $2n/3$, then G contains a hamiltonian square cycle.*

We shall say that a square cycle of length greater than $2n/3$ in a graph G on n vertices is a *long* square cycle. Thus Theorem 0.3 says that if a graph on n vertices with $\delta(G) \geqslant 2n/3$ has a long square cycle, then it has a hamiltonian square cycle. A square cycle of length exactly $2n/3$ in G is called a *critical* square cycle.

Before proceeding, we should note that Komlós et al. [7] have just used the Regularity Lemma and a new lemma called the Blow-Up Lemma to prove that Pósa's conjecture is true if n is sufficiently large. This is a beautiful result that uses very powerful techniques, but unfortunately 'sufficiently large' is considerably larger than 2^{100}, and so the result does not apply to any 'ordinary' graphs.

A subgraph of a graph G is called a clique if it is complete, i.e., any two vertices are adjacent. A k-clique is a clique on k vertices. The clique number of G is the largest k such that G has a k-clique. A maximal clique is a clique that is not properly contained in any other clique in G. The main result of this paper is the following theorem.

Theorem 0.4. *Let G be a graph on n vertices with $\delta(G) \geqslant 2n/3$. If G contains a maximal 4-clique, then G contains a hamiltonian square cycle.*

While we have checked the theorem for all values of $n \leqslant 15$ by ad hoc methods, here we shall only give the proof for $n \geqslant 16$.

The hypothesis that G contains a maximal 4-clique may seem somewhat unnatural. We offer the following justifications. As we have seen, the extremal example of a graph on n vertices with minimum degree $2n/3 - 1/3$ that does not contain a hamiltonian square cycle has clique number 3. It is easy to prove that a graph $G = (V, E)$ with a maximal 3-clique $K = \{v_1, v_2, v_3\}$ and $\delta(G) \geqslant 2n/3$ has a hamiltonian square cycle: For $i \in \{1,2,3\}$, let $V_i = \{x \in V: x$ is not adjacent to $v_i\}$. Since K is maximal,

$V = V_1 \cup V_2 \cup V_3$. By the degree condition, $|V_i| \leqslant n/3$, for all i. Thus $|V_i| = n/3$ and $V_i \cap V_j = \emptyset$, for all $i \neq j$. So G is the complete tripartite graph with $n/3$ vertices in each part and so has a square hamiltonian cycle. From this point of view it is natural to ask what happens if G has a maximal 4-clique K and $\delta(G) \geqslant 2n/3$. Then $G - K$ satisfies the hypothesis of Theorem 0.2 and thus contains a hamiltonian square path P. The proof of Theorem 0.4 involves using P and K to find a long square cycle in G. Then by Theorem 0.3, G has a hamiltonian square cycle. In terms of attacking Pósa's conjecture, it seems that knowing that every 4-clique can be extended to a 5-clique could be a useful tool. Finally, we believe that Theorems 0.2 and 0.3 are very likely to be useful in proving Pósa's conjecture. Our use of these theorems in the proof of Theorem 0.4, while not directly applicable to the conjecture, provides some evidence for this claim.

The proof of Theorem 0.4 is organized as follows. In Section 1 we present some easy combinatorial facts. In Section 2 we consider a graph G satisfying the hypothesis of the theorem. The vertices of G are partitioned in a special way and various technical facts about the partition are proved. Finally, in Section 3 the preliminary results of Sections 1 and 2 are used to prove the theorem. In the remainder of this section we explain our notation.

Let X and Y be two sets of vertices of a graph G. We denote by $d(X;Y)$ the number of edges with one endpoint in X and the other endpoint in Y. We shall also write $d(w,x;Y)$ for $d(\{w,x\};Y)$, $d(x;Y)$ for $d(\{x\};Y)$, etc. The set of vertices adjacent to a vertex x is denoted by $N(x)$. Also $N_Y(x) = N(x) \cap Y$. If P and Q are sequences of vertices, in particular paths, we denote P followed by Q by $P+Q$. If P is a sequence and X is a set, $P+X$ denotes P followed by some ordering of X. For example, if P is a square path (cycle) and X is a set of vertices, we shall say that $P+X$ is a square path (cycle), when we mean that for some sequence Q of the vertices of X, $P+Q$ is a square path (cycle).

1. Tools

In this section we present some easy combinatorial results. Fix a graph G.

Proposition 1.1. *Let $s_1, s_2, \ldots, s_i$ be nonnegative integers less than 4 such that* (i) $\sum_{1 \leqslant h \leqslant i} s_h \geqslant 2i$, (ii) $s_h + s_{h+1} \leqslant 4$, *for all $h < i$, and* (iii) $s_i \leqslant 2$. *Then $s_h \geqslant 2$ and $s_h + s_{h+1} = 4$, for all odd $h < i$. Moreover, if $s_i = 1$, then i is even and for all odd $h < i$, both $s_h = 3$ and $s_{h+1} = 1$. Also if for some $h \leqslant i$, $s_h = 2$, then for all g such that $h \leqslant g \leqslant i$, $s_g = 2$.*

Proposition 1.2. *Let $C = (z_1, \ldots, z_c)$ be a cycle of length c in G. Let x be a vertex not in C and let d and s be positive integers. If $d(x;C) > dc/s$, then there exists an integer a such that $d(x; z_{a+1}, \ldots, z_{a+s}) > d$, where addition of indices is modulo c.*

Proposition 1.3. *Let $P=(z_1,\ldots,z_p)$ be a path of length p in G. Let x be a vertex not in P. If $d(x;P)>dp/s+d/s$, then there exists an integer a such that $d(x;z_{a+1},\ldots,z_{a+s})>d$.*

Proposition 1.4. *Let $C=(z_1,\ldots,z_c)$ be a square cycle of length c in G. Let x be a vertex not in C. If x is adjacent to four consecutive vertices z_{a+1}, z_{a+2}, z_{a+3}, and z_{a+4} of C, then $(\ldots z_{a+1},z_{a+2},x,z_{a+3},z_{a+4},\ldots)$ is a square cycle of length $c+1$.*

Lemma 1.5. *Let $C=(z_1,\ldots,z_c)$ be a square cycle of length $c\geqslant 12$ in G. Let xy be an edge of G such that neither x nor y is a vertex of C. If $d(x,y;C)\geqslant 3c/2-1$, then $C+\{x,y\}$ contains a square cycle of length at least $c+1$.*

Proof. Without loss of generality, assume $d(x;C)\leqslant d(y;C)$. By Proposition 1.4, we may assume that no vertex in $V-C$ is adjacent to four consecutive vertices of C. Thus by Proposition 1.2, $3c/4-1\leqslant d(x;C)\leqslant d(y;C)\leqslant 3c/4$.

Claim 1. *Suppose that x is adjacent to 3 consecutive vertices z_{a+1}, z_{a+2}, and z_{a+3} of C. If any of the following hold, then there exists a square cycle longer than c.*

(i) *y is adjacent to each of z_{a+2}, z_{a+3}, and z_{a+4} (or z_a, z_{a+1}, and z_{a+2});*
(ii) *y is adjacent to each of z_{a+3}, z_{a+4}, and z_{a+5} (or z_{a-1}, z_a, and z_{a+1});*
(iii) *y is adjacent to each of z_{a+3}, z_{a+5}, and z_{a+6} (or z_{a-2}, z_{a-1}, and z_{a+1}).*

Proof. If (i) holds, then $(\ldots z_{a+1},z_{a+2},x,y,z_{a+3},z_{a+4},\ldots)$ is a square cycle of length $c+2$. If (ii) holds, then $(\ldots z_{a+1},z_{a+2},x,z_{a+3},y,z_{a+4},z_{a+5},\ldots)$ is a square cycle of length $c+2$. If (iii) holds, then $(\ldots z_{a+1},z_{a+2},x,z_{a+3},y,z_{a+5},z_{a+6},\ldots)$ is a square cycle of length $c+1$.

Case 1: *$d(x;C)=3c/4-1$ and $d(y;C)=3c/4$.* Then y is nonadjacent to exactly every fourth vertex of C. First suppose $c\geqslant 15$. Then $3c/4-1>2c/3$. So by Proposition 1.2, x is adjacent to three consecutive vertices z_{a+1}, z_{a+2}, and z_{a+3} of C. Thus either (i)–(iii) must hold. Otherwise $c=12$ and x is adjacent to every third vertex of C. It is easy to check that there exists $z\in C$ such that neither x nor y is adjacent to z. Then $(C-z)+(x,y)$ is a square cycle of length $c+1$.

Case 2: $d(x;C)=3c/4-1/2=d(y;C)$. First suppose $c\geqslant 15$. Then $3c/4-1/2>5c/7$. By Proposition 1.2, for some a, x is adjacent to each of the vertices z_{a+1}, z_{a+2}, z_{a+3}, z_{a+5}, z_{a+6}, and z_{a+7}. If Q is a segment of C of length q such that $3q>4d(y;Q)+5$, then $3c/4-1/2-d(y;Q)=d(y;C-Q)>3(c-q)/4+3/4$. Thus y is adjacent to four consecutive vertices in $C-Q$. So we are done if y is adjacent to at most d of q consecutive vertices, where either (iv) $d=0$ and $q=2$, (v) $d=2$ and $q=5$, or (iv) $d=6$ and $q=10$. If y is not adjacent to z_{a+4} then by (iv) we may assume that y is adjacent to z_{a+3} and z_{a+5}. Thus by (iii) we may assume that y is nonadjacent to z_{a+2} and z_{a+6}. But then y is adjacent to at most two of five consecutive vertices and we are done by (v). So suppose that y is adjacent to z_{a+4}. Then y is not adjacent to at least one of z_{a+3} and z_{a+5} by (ii). By symmetry, assume y is not adjacent to z_{a+3}.

Then by (i) and (ii) y is also nonadjacent to at least one vertex in each of the triples (z_a, z_{a+1}, z_{a+2}), $(z_{a+4}, z_{a+5}, z_{a+6})$, $(z_{a+7}, z_{a+8}, z_{a+9})$. Thus we are done by (vi). Otherwise $c = 14$. Then x is nonadjacent to four vertices of C. These four vertices partition the neighbors of x in C into segments of lengths (1,3,3,3), (2,2,3,3), or (2,3,2,3) and the same holds for y. Suppose x and y are nonadjacent to the same vertex z. It is easy to check that there exists a vertex z_a adjacent to neither x nor y such that x is adjacent to z_{a-2}, z_{a-1}, and z_{a+1} and y is adjacent to z_{a-1}, z_{a+1}, and z_{a+2} (or vice versa). But then $(C - x) + xy$ is a square cycle of length $c + 1$. So x and y are not nonadjacent to the same vertex. There exists a vertex z_a such that x is adjacent to z_{a+1}, z_{a+2}, and z_{a+3} and y is adjacent to z_{a+3} (or z_{a+1}). Then x is adjacent to neither z_a nor z_{a+4} and so y is adjacent to both z_a and z_{a+4}. By (i) and (ii) we may assume that y is adjacent to neither z_{a+2} nor z_{a+5}. So y is adjacent to z_{a+1} and by (ii) we may assume that y is not adjacent to z_{a-1}. Thus y has the pattern (2,2,3,3) and by symmetry, so does x. It follows that we are done by (i) applied to either $(z_{a+5}, z_{a+6}, z_{a+7})$ or $(z_{a-3}, z_{a-2}, z_{a-1})$. □

Lemma 1.6. *Let $C = (z_1, \ldots, z_c)$ be a square cycle of length c with z_{c-3} adjacent to z_c. Suppose v is a vertex such that $v \notin C$, v is not adjacent to z_{c-3}, and $d(v; C) \geqslant 3|C|/4$. Then $C \cup \{v\}$ contains a hamiltonian square cycle.*

Proof. If $d(v; C) > 3|C|/4$ then we are done by Proposition 1.4. Otherwise equality holds and v is nonadjacent to exactly every fourth vertex of C. In particular, v is adjacent to each of the vertices z_{c-6}, z_{c-5}, and z_{c-4}. Thus $(z_1, \ldots, z_{c-6}, z_{c-5}, v, z_{c-4}, z_{c-2}, z_{c-3}, z_{c-1}, z_c)$ is a square cycle. □

2. The partition

For the rest of the paper we will use the following notation. Let G be a graph on n vertices with minimum degree $\delta(G) \geqslant 2n/3$. Let K be a maximal 4-clique in G. Since K is maximal, each vertex in $G - K$ is adjacent to at most three vertices in K. Thus $G - K$ is a graph on $n - 4$ vertices with minimum degree $\delta(G - K) \geqslant 2n/3 - 3 = (2(n-4) - 1)/3$. By Theorem 0.2, $G - K$ contains a hamiltonian square path $P = (z_1, \ldots, z_p)$.

Lemma 2.1. *Let $Q = (z_{s+1}, \ldots, z_{s+q})$ be a segment of P such that $d(z_{s+q}; K) = 3$.*

(1) *If $d(z_{s+1}; K) = 3$ and there exist distinct vertices x and y such that $x \in N_K(z_{s+1}) \cap N_K(z_{s+2})$ and $y \in N_K(z_{s+q-1}) \cap N_K(z_{s+q})$, then $Q + (N_K(z_{s+1}) \cup N_K(z_{s+q}))$ is a square cycle.*

(2) *If $d(z_{s+1}; K) \geqslant 2$ and $N_K(z_{s+1}) \cap N_K(z_{s+2}) \neq \emptyset$ and $|N_K(z_{s+q-1}) \cap N_K(z_{s+q})| \geqslant 2$, then there exists a path S in K such that $|S| \geqslant 2$ and $Q + S$ is a square cycle.*

In most applications of either part of Lemma 2.1 we will check the hypothesis on $N_K(z_{s+q-1}) \cap N_K(z_{s+q})$ by observing that $d(z_{s+q-1}; K) = 3$.

Lemma 2.2. *Let $Q=(z_{s+1},\dots,z_{s+q})$ be a segment of P. Let $xy\in E(K)$ be an edge such that $Q+(x,y)$ is a square cycle. Let $\{v,w\}=K-\{x,y\}$. If $d(z_{s+q};K)=3$ and $12\leqslant q+2\leqslant 2n/3$, then G contains a square cycle of length at least $q+3$.*

Proof. Let $X=(z_1,\dots,z_s)$ and $Y=(z_{s+q+1},\dots,z_p)$. If $h\leqslant s, d(z_h;vw)=2$, and $d(z_{h+1};vw)\geqslant 1$, then $\{v,w\}+(z_h,\dots,z_{s+q},x,y)$ is a square cycle of length at least $q+5$. So assume that if $h\leqslant s$ and $d(z_h;vw)=2$, then $d(z_{h+1};vw)=0$. Similarly we can assume that if $h\geqslant s+q+1$ and $d(z_h;vw)=2$, then $d(z_{h-1};vw)=0$. Since $d(z_{s+q};K)=3$, $d(z_{s+q};vw)\geqslant 1$ and thus $d(z_{s+q+1};vw)\leqslant 1$. We conclude that $d(X;vw)\leqslant s+1$ and $d(Y;vw)\leqslant p-s-q$. Thus

$$\begin{aligned}d(vw;Q+xy) &\geqslant 4n/3-d(vw;(P-Q)\cup\{v,w\})\geqslant 4n/3-((n-4)-q+1)-2\\ &\geqslant n/3+q+1\geqslant 3(q+2)/2-1.\end{aligned}$$

Using Lemma 1.5, since $q+2\geqslant 12$, $Q\cup K$ contains a square cycle of length at least $q+3$. $\square$

Lemma 2.3. *Let $Q=(z_{s+1},\dots,z_{s+q})$ be a segment of P of length $q\geqslant 10$. If $d(z_{s+1};K)\geqslant 2$, $d(z_{s+q};K)=3$, $N_K(z_{s+1})\cap N_K(z_{s+2})\neq\emptyset$, and $|N_K(z_{s+q-1})\cap N_K(z_{s+q})|\geqslant 2$, then G contains a square cycle of length at least $q+3$.*

Proof. Use Lemmas 2.1(2) and 2.2. $\square$

Partition P into segments as follows. Let i and j to be the maximum integers such that $d(K;I)\leqslant 2i$ and $d(K;J)\leqslant 2j$. Choose a and b to be the least integers such that $d(z_{i+a+1};K)=3=d(z_{i+a+2};K)$ and $d(z_{p-j-b};K)=3=d(z_{p-j-b-1};K)$. Notice that any of i,j,a, or b may be zero and thus any of I,J,A, or B may be empty. Now set

$$\begin{aligned}&I=(z_1,\dots,z_i),\\ &A=(z_{i+1},\dots,z_{i+a})=(\alpha_1,\dots,\alpha_a),\\ &M=(z_{i+a+1},\dots,z_{i+a+m})=(\mu_1,\dots,\mu_m),\\ &B=(z_{p-j-b+1},\dots,z_{p-j})=(\beta_b,\dots,\beta_1),\end{aligned}$$

and

$$J=(z_{p-j+1},\dots,z_p)=(\varphi_j,\dots,\varphi_1).$$

This completes the definition of the partition. Without loss of generality, assume that $a\geqslant b$. The following proposition is an easy consequence of the above construction.

Proposition 2.4. (1) *$d(I;K)=2i$ and $d(J;K)=2j$;*
(2) *$d(\mu_1;K)=d(\mu_2;K)=d(\mu_{m-1};K)=d(\mu_m;K)=3$;*
(3) *if $a\neq 0$, then $d(\alpha_1;K)=3$, and if $b\neq 0$, then $d(\beta_1;K)=3$;*
(4) *if $a\neq 0$, then $d(\alpha_a;K)\leqslant 2$, and if $b\neq 0$, then $d(\beta_b;K)\leqslant 2$;*

(5) *if* $a \neq 0$, *then* $a \geqslant 2$, *and if* $b \neq 0$, *then* $b \geqslant 2$;

(6) *if* $a \neq 0$, *then* $d(\alpha_2; K) = 2$, *and if* $b \neq 0$, *then* $d(\beta_2; K) = 2$;

(7) $d(A; K) \leqslant 5a/2$, *with equality only if* a *is even and, for every odd* $h < a$, $d(\alpha_h;\ K) = 3$ *and* $d(\alpha_{h+1}; K) = 2$; *and* $d(B; K) \leqslant 5b/2$, *with equality only if* b *is even and, for every odd* $h < b$, $d(\beta_h; K) = 3$ *and* $d(\beta_{h+1}; K) = 2$;

(8) $d(M; K) \leqslant 3m$.

Lemma 2.5. *If* G *does not contain a long square cycle, then*

(1) $2n/3 - 4 + 4(\lceil 2n/3 \rceil - 2n/3) + (a-b)/2 + 5a/2 - d(K; A) + 5b/2 - d(K; B) \leqslant a + m \leqslant 2n/3 - 3$.

(2) $2n/3 - 4 \leqslant a + m \leqslant 2n/3 - 3$ *and* $n = 0 \bmod 3$;

(3) *if* $a + m = 2n/3 - 4$, *then* $a = b$, a *is even, and for every odd* $h < b$, $d(\beta_h; K) = 3$.

Proof. Let $r = \lceil 2n/3 \rceil - 2n/3$. Using Proposition 2.4(1), (7), (8), we have

$$4(2n/3 - 3 + r) \leqslant d(K; P) \leqslant d(K; I \cup A \cup M \cup B \cup J),$$

$$\begin{aligned} 8n/3 + 4r - 12 &\leqslant 2i + 2j + 5a/2 + 5b/2 + 3m - 5a/2 \\ &\quad + d(K; A) - 5b/2 + d(K; B) \\ &\leqslant 2(n-4) + a + m - (a-b)/2 - 5a/2 \\ &\quad + d(K; A) - 5b/2 + d(K; B). \end{aligned}$$

(i), $2n/3 - 4 + 4r + 5a/2 - d(K; A) + 5b/2 - d(K; B) + (a-b)/2 \leqslant a + m$. By Proposition 2.4(2)–(4) and Lemma 2.1(1). G contains a square cycle of length $a + m + 3$. Since G does not contain a long square cycle (ii) $a + m + 3 \leqslant 2n/3$. Combining (i) and (ii) yields (1). Either $4r = 0$ or $4r > 1$. By Proposition 2.4(7), $5a/2 - d(K; A) \geqslant 0$ and $5b/2 - d(K; B) \geqslant 0$. By the choice of a, $(a-b)/2 \geqslant 0$. Thus (2) holds. If $a + m = 2n/3 - 4$, then $a = b$ and $5b/2 - d(K; B) = 0$. So (3) follows by Proposition 2.4(7). □

Lemma 2.6. *If* G *does not contain a long cycle and* $a + m = 2n/3 - 3$, *then*:

(1) $d(z_t, z_{t+1}; K) \leqslant 4$, *where* $t = i + a + m + 1$; ($z_t = \beta_b$ *or* $z_t = \varphi_j$) *and*

(2) *if* $b > 0$, *then* $a - b \leqslant 1$, b *is odd*, $d(\beta_h; K) = 3$ *if* h *is odd and* $h < b$, $d(\beta_h; K) = 2$ *otherwise*, $d(\alpha_h; K) = 3$ *if* h *is odd and* $h < a$, *and* $d(\alpha_h; K) = 2$ *otherwise.*

Proof. Suppose (1) fails. We obtain a contradiction by showing that G contains a long square cycle. If $d(z_t; K) = 3$, then by Lemma 2.1(1), $(z_{i+1}, \ldots, z_t) + K$ contains a square cycle of length $a + m + 1 + 3 > 2n/3$. So suppose $d(z_t; K) = 2$ and $d(z_{t+1}; K) = 3$. Then $N_K(z_{i+1}) = N_K(\mu_m)$, since otherwise, by Lemma 2.1(1), $(z_{i+1}, \ldots, \mu_m) + K$ is a square cycle of length $a + m + 4 > 2n/3$. Say $N_K(z_{i+1}) = \{w, x, y\}$, $y \sim z_{i+2}$, and v is the remaining element of K. We shall consider three cases.

Case 1: $v \notin N(z_t)$. Then $C = (y, z_{i+1}, \ldots, z_t) + (N_K(z_t) - \{y\})$ is a square cycle of length at least $2n/3$. If $|C| = 2n/3$, then by Lemma 2.2 (in reverse order), G contains a long square cycle.

Case 2: $v \notin N(z_{t+1})$. Then $N_K(z_{t+1}) = \{w, x, y\}$. One of w and x, say w, is adjacent to μ_{m-1}. Then $(z_{i+1}, \ldots, \mu_m, w, z_{t+1}, x, y)$ is a long square cycle.

Case 3: $v \in N(z_t) \cap N(z_{t+1})$. Then $(y, z_{i+1}, \ldots, z_{t+1}, v) + \{x, y\}$ is a long square cycle.

So (1) holds. Suppose $b > 0$. Then by (1) and Proposition 2.4(7), $5b/2 - d(K; B) > 0$. Thus by Lemma 2.5, $a - b \leqslant 1$. Moreover, if $a - b = 1$, $d(K; A) = 5a/2$. Thus a is even and b is odd. If $a - b = 0$, then by (1) and symmetry, $5a/2 - d(K; A) > 0$. Thus $5a/2 - d(K; A) = 1/2 = 5b/2 - d(B; K)$. Thus b is odd. Finally $d(B - \{\beta_{b-1}, \beta_b\}; K) = 5b/2 - 9/2$. Since for all $h < b$, $d(\beta_h, \beta_{h+1}; K) \leqslant 5$, (2) follows. □

Lemma 2.7. *Suppose that G does not contain a long square cycle. Then*:

(1) *for all* $h < i$, $d(z_h, z_{h+1}; K) \leqslant 4$; *and*

(2) $d(u; I - \{z_i\}) \leqslant \lceil (i-1)/2 \rceil$, *for all* $u \in K$.

(3) *If* $d(z_i; K) = 1$, *then* i *is even.*

If $d(z_i; K) \leqslant 2$, *then*:

(4) $d(u; I) \leqslant \lceil i/2 \rceil$;

(5) *for all odd* $h < i$, $d(z_h; K) \geqslant 2$ *and* $d(z_h, z_{h+1}; K) = 4$; *and*

(6) *for all* $h < i$, *if* $d(z_h; K) = 2$, *then* $d(z_{h+1}; K) = 2$ *and* $N_K(z_h) = K - N(z_{h+1})$.

Proof. Note that (∗) for all $h < i$ and all $u \in K$, if $d(z_h; K) \geqslant 2$, then $d(z_h, z_{h+1}; u) \leqslant 1$. Otherwise by Lemma 2.3, applied to $(z_h, z_{h+1}, \ldots, \mu_{m-1}, \mu_m)$ shows that G contains a square cycle of length at least $2 + a + m + 3 > 2n/3$. Thus (1) holds. Now assume that $d(z_i; K) \leqslant 2$. Using (1) and Proposition 1.1, (3) and (5) hold. So (∗∗) for every odd $h < i$, $d(u; z_h, z_{h+1}) \leqslant 1$. Thus (4) holds. Using (∗), (1), and (5), (6) holds.

It remains to prove (2), which follows from (∗∗), unless $d(z_i; K) = 3$. If for all even $h \leqslant i$, $d(z_{i-h}; K) = 3$, then for all even h with $2 \leqslant h < i$, $d(u; z_{i-h}, z_{i-h+1}) \leqslant 1$. So $d(u; I - \{z_i\}) \leqslant \lceil (i-1)/2 \rceil$. Otherwise choose the maximum g with $1 \leqslant g < i/2$ such that $d(z_{i-2g}; K) = 3$. Then $d(u; z_{i-2g}, \ldots, z_{i-1}) \leqslant g$. Also, $d(z_{i-2g-1}; K) \leqslant 1$. Thus $d(z_{i-2g-1}, \ldots, z_i; K) \leqslant 4g + 4$ and $d(z_1, \ldots, z_{i-2g-2}; K) \geqslant 2i - 4g - 4$. By the choice of $g, d(z_{i-2g-2}; K) \leqslant 2$. Thus by Proposition 1.1, for every odd $h < i - 2g - 2$, $d(z_h; K) \geqslant 2$ and thus $d(u; z_h, z_{h+1}) \leqslant 1$.

Case 1: $d(z_{i-2g-2}; K) = 1$. Then by Proposition 1.1, $i - 2g - 2$ is even. It follows that $d(u; z_1, \ldots, z_{i-2g-2}) \leqslant (i - 2g - 2)/2$. Thus $d(u; I - \{z_i\}) = d(u; z_1, \ldots, z_{i-2g-2}) + d(u; z_{i-2g-1}) + d(u; z_{i-2g}, \ldots, z_{i-1}) \leqslant (i - 2g - 2)/2 + 1 + g \leqslant \lceil (i-1)/2 \rceil$.

Case 2: $d(z_{i-2g-2}; K) = 2$. Then $d(u; z_{i-2g-2}, z_{i-2g-1}) \leqslant 1$. Thus $d(u; I - \{z_i\}) = d(u; z_1, \ldots, z_{i-2g-3}) + d(u; z_{i-2g-2}, z_{i-2g-1}) + d(u; z_{i-2g}, \ldots, z_{i-1}) \leqslant \lceil (i - 2g - 3)/2 \rceil + 1 + g \leqslant \lceil (i-1)/2 \rceil$. □

Lemma 2.8. *Suppose that G does not contain a long square cycle. Then*

(1) *for all* $h < j$, $d(\varphi_h, \varphi_{h+1}; K) \leqslant 4$; *and*

(2) $d(u; J - \{\varphi_j\}) \leqslant \lceil (j-1)/2 \rceil$, *for all* $u \in K$.

(3) *If* $d(\varphi_j; K) = 1$, *then* j *is even.*

(4) *If* $d(\varphi_j; K) \leqslant 2$, *then* $d(u; J) \leqslant \lceil j/2 \rceil$.

If $a + m = 2n/3 - 4$ *and* $d(\varphi_j; K) \leqslant 2$, *then*

(5) *for all odd* $h < j$, $d(\varphi_h; K) \geqslant 2$ *and* $d(\varphi_h, \varphi_{h+1}; K) = 4$;

(6) *for all* $h < j$, *if* $d(\varphi_h; K) = 2$, *then* $d(\varphi_{h+1}; K) = 2$ *and* $N_K(\varphi_h) = K - N(\varphi_{h+1})$.

Proof. Assume that $a + m = 2n/3 - 3$, since otherwise the lemma follows from Lemma 2.7 and symmetry. In particular, (5) and (6) hold. Then $d(\varphi_j; K) \leqslant 2$, since otherwise by Lemma 2.1(1), $(z_{i+1}, \ldots, z_{p-j}, \varphi_j) \cup K$ contains a square cycle of length at least $a + m + b + 1 + 3 > 2n/3$. We first show (1). Otherwise there exists $h < j$ and $u \in K$ such that $d(\varphi_h, \varphi_{h+1}; K) > 4$, $d(\varphi_h; K) \geqslant 2$ and u is adjacent to both φ_h and φ_{h+1}. Then by Proposition 2.4(2) and Lemma 2.3, $(\mu_1, \mu_2, \ldots, \varphi_{h+1}, \varphi_h) + K$ contains a square cycle of length at least $m + b + 2 + 3 = m + a - (a - b) + (j - h + 1) + 3 \leqslant 2n/3$. Thus $a - b = 2$ and $h = j - 1$. Since $d(\varphi_{j-1}, \varphi_j; K) > 4$, we must have $d(\varphi_{j-1}; K) = 3$. So by Lemma 2.6(1), $b > 0$, which contradicts Lemma 2.6(2). So (1) holds. Thus using Proposition 1.1,(3) holds and, for every odd $h < j$, $d(\varphi_h; K) \geqslant 2$. So for every odd $h < j$ and every $u \in K$, $d(u; \varphi_h, \varphi_{h+1}) \leqslant 1$. Thus (2) and (4) hold. □

Lemma 2.9. *Suppose that G does not contain a long square cycle. If* $u \in K$ *and* u *is not adjacent to* α_1, *then* $d(u; B) \leqslant \lceil b/2 \rceil$.

Proof. It suffices to show that for every odd $h < b$, $d(u; \beta_{h+1}, \beta_h) \leqslant 1$. So suppose that h is a counter example. By Lemmas 2.5(3) and 2.6(2), $d(\beta_h; K) = 3$. Thus by Lemma 2.1(1), $(\alpha_1, \ldots, \beta_h) + K$ contains a cycle of length at least $a + m + 2 + 3 > 2n/3$, which is a contradiction. □

3. The proof of Theorem 0.4.

As mentioned in the introduction, we will only prove the case $n \geqslant 16$. By Theorem 0.3, it suffices to show that G contains a long cycle. Assume this is false. First suppose that $a + m = 2n/3 - 3$. By Lemma 2.1(1) and Proposition 2.4(2)–(4), there exists a path $S \subset K$ such that $|S| \geqslant 3$ and $C = (\alpha_1, \ldots, \mu_m) + S$ is a square cycle. Since C is not long, $|S| = 3$. So again by Lemma 2.1(1), $N_K(\alpha_1) = N_K(\mu_m)$. Say $S = (w, x, y)$ and v is the remaining vertex of K. Since μ_m is adjacent to y, but not v, by Lemma 1.6, it suffices to show that $d(v; C) \geqslant 3|C|/4 = n/2$. If $d(z_i; K) \geqslant 2$, then, using Lemma 2.3, with $Q = (z_i, \alpha_1, \ldots \mu_m)$, G contains a square cycle of length at least $1 + a + m + 3 > 2n/3$. So assume $d(z_i; K) \leqslant 1$. By Lemma 2.7(3.4) $d(v; I) \leqslant i/2$. It suffices to show that $d(v; B \cup J) \leqslant (b + j + 1)/2$, for then $d(v; C) \leqslant 2n/3 - d(v; I \cup B \cup J) \leqslant 2n/3 - (i + b + j + 1)/2 = n/2$.

Note that $d(\varphi_j; K) \leqslant 2$; since otherwise by Lemma 2.3, with $Q = (\alpha_1, \ldots, \varphi_j)$, G contains a square cycle of length at least $a + m + b + 1 + 3 > 2n/3$. So by Lemma 2.8(2), $d(v; J) \leqslant \lceil j/2 \rceil$. By Lemma 2.9, $d(v; B) \leqslant (b + 1)/2$. So if either j or b is even, then $d(v; B \cup J) \leqslant (b + j + 1)/2$. So suppose that both j and b are odd. By Proposition 2.4(5), $b \geqslant 2$. Since $a \geqslant b$, $a \geqslant 3$. By Lemma 2.6(2), $d(\alpha_1; K) = 3 = d(\alpha_3; K)$, and $d(\alpha_2, K) = 2$.

It suffices to show that (i) for all $h<b$, $d(v;\beta_{h+1},\beta_h)\leqslant 1$ and (ii) $d(v,\beta_1,\varphi_j)\leqslant 1$, for then, using Lemma 2.8(2),

$$\begin{aligned} d(v;B\cup J) &\leqslant d(v;(J-\{\varphi_j\})\cup\{\varphi_j,\beta_1\}\cup(B-\{\beta_1\})) \\ &\leqslant (j-1)/2+1+(b-1)/2\leqslant (b+j)/2. \end{aligned}$$

For (i) suppose that for some $h<b$, $d(v;\beta_{h+1},\beta_h)=2$. If $d(\beta_h;K)=3$, then $(\alpha_1,\ldots,\beta_h)+K$ contains a square cycle of length at least $a+m+2+3>2n/3$ by Lemma 2.1(1). So assume, using Lemma 2.6(2), that $d(\beta_h;K)=2$, $d(\beta_{h+1};K)\geqslant 2$, and $d(z_{p-j+1-h-2},\beta_{h+1};K)=5$. (If $d(\beta_{h+1};K)=2$, then $h+1=b$ and $z_{p-j+1-h-2}=\mu_m$.) If β_h is not adjacent to y, then $(y,\alpha_1,\ldots,\beta_h,v)+\{w,x\}$ is a square cycle of length at least $a+m+2+4>2n/3$. If β_{h+1} is not adjacent to y, then $(y,\alpha_1,\ldots,\beta_{h+1})+\{v,w,x\}$ is a square cycle of length at least $a+m+1+4>2n/3$. So suppose that y is also adjacent to both β_h and β_{h+1}. Then by Lemma 2.3 with $Q=(\alpha_2,\ldots,\beta_h)$, G contains a square cycle of length at least $a-1+m+2+3>2n/3$. So (i) holds. The proof of (ii) is analogous after first noting that $d(\varphi_j;K)=2$, by Lemma 2.8(3), and both $d(\beta_1;K)=3$ and $d(\beta_2;K)=2$ by, Proposition 2.4(3), (6).

So $a+m=2n/3-4$. Suppose that either $d(z_i;K)=3$ or $d(\varphi_j;K)=3$. Without loss of generality, $d(z_i;K)=3$. By Lemma 2.1(1) and Proposition 2.4(2), (3), (6), there exists a path $S\subset K$ such that $|S|\geqslant 3$ and $C=(z_i,\alpha_1,\alpha_2,\ldots,\mu_{m-1},\mu_m)+S$ is a square cycle. Since C is not long, $|S|=3$. So using Lemma 2.1(1), $S=(w,x,y)=N_K(z_i)=N_K(\mu_m)$. Let v be the remaining vertex of K. Since μ_m is adjacent to y, but not v, by Lemma 1.6, it suffices to show that $d(v;C)\geqslant 3|C|/4$. By Lemma 2.7(2), $d(v;I-\{z_i\})\leqslant\lceil(i-1)/2\rceil$. If $d(\varphi_j;K)\geqslant 2$, then $d(z_{p-j},\varphi_j;K)\geqslant 5$. Thus using Lemma 2.3, with $Q=(z_i,\alpha_1,\alpha_2,\ldots,z_{p-j-1},\varphi_j)$, G contains a square cycle of length at least $1+a+m+1+3>2n/3$. Otherwise by Lemma 2.8(3), (4), $d(v;J)\leqslant j/2$. By Lemma 2.9, $d(v,B)\leqslant b/2$. Thus $d(v;C)\geqslant 2n/3-d(v;(I-\{z_i\})\cup J\cup B)\geqslant 2n/3-(i+j+b)/2=n/2=3|C|/4$.

So $d(z_i;K)\leqslant 2$ *and* $d(\varphi_j;K)\leqslant 2$. Suppose that either i is odd or j is odd. Without loss of generality, we may assume that i is odd. By Lemma 2.7(3), $d(z_i;K)=2$. Using Lemma 2.1(2), choose a maximum path $S\subset K$ such that $C=(z_i,\alpha_1,\ldots,\mu_m)+S$ is a square cycle. Then $|S|\geqslant 2$. Since C is not long, $|S|\leqslant 3$. Consider any $w\in K-S$. By Lemma 2.7(2) $d(w;I-\{z_i\})\leqslant(i-1)/2$. By Lemma 2.9 $d(w;B)\leqslant b/2$. By Lemma 2.8 (4) $d(w;J)\leqslant\lceil j/2\rceil$.

First suppose that $S=(w,x,y)$. Let v be the unique element of $K-S$. Note that v is not adjacent to μ_m, since otherwise $S=(w,\mu_m,x,y)$ gives a longer cycle. Since $d(\mu_m;K)=3$, μ_m is adjacent to y. Finally,

$$\begin{aligned} d(v;C) &\geqslant 2n/3-d(v;(I-\{z_i\})\cup B\cup J) \\ &\geqslant 2n/3-(i-1+b+j+1)/2\geqslant n/2=3c/4. \end{aligned}$$

Thus by Lemma 1.6 G contains a long square cycle.

So suppose that $S=(x,y)$. Note that μ_{m-1} is adjacent to y since otherwise we can choose S to have cardinality 3. If j is even, then $d(w;J)\leqslant j/2$, for any $w\in K-S$. Otherwise $d(\varphi_j;K)=2$, by Lemma 2.8(3). If $N_K(z_i)\cap N_K(\varphi_j)=\emptyset$, then $(z_i,\ldots,z_{p-j-1},\varphi_j)+K$

is a square cycle of length $1+a+m+b+1+4>2n/3$. Else there exists $w \in K$ such that w is adjacent to neither z_i nor φ_j. Thus $w \notin S$. Using Lemma 2.8(2), $d(w;J)=d(w;J-\{\varphi_j\})\leqslant j/2$. By Lemma 2.5(3), b is even. So regardless of the parity of j, we have

$$\begin{aligned} d(w;C) &\geqslant 2n/3 - d(w;(I-\{z_i\})\cup B\cup J\cup(K-S)) \\ &\geqslant 2n/3-(i-1+b+j)/2-1 \\ &\geqslant n/2-1/2>n/2-3/4=3c/4. \end{aligned}$$

Thus by Lemma 1.4, $C'=C\cup\{w\}$, contains a critical square cycle. Let v be the remaining vertex of K. Note that since $d(z_i;K)=2$, z_i is not adjacent to v. As above

$$\begin{aligned} d(v;C') &\geqslant 2n/3-d(v;(I-\{z_i\})\cup B\cup J) \\ &\geqslant 2n/3-(i-1+b+j+1)/2=n/2. \end{aligned}$$

We are done by Proposition 1.4, unless equality holds. In this case, v is nonadjacent to exactly every fourth vertex of C'. By the maximality of S, $(\mu_{m-1},\mu_m,x,y,z_i)$ is a segment of C'. Thus v is not adjacent to μ_{m-1} and so μ_{m-1} is adjacent to y. Thus we are done by Lemma 1.6.

So both i and j are even. We first show that $A\cup M\cup K$ contains a critical cycle D. Otherwise, by Lemma 2.1(1), there exists a path $(w,x,y)\subset K$ such that $C=A+M+(w,x,y)$ is a square cycle and $N_K(\alpha_1)=N_K(\mu_m)=\{w,x,y\}$. Let v be the remaining element of K. By Proposition 1.2 it suffices to show that $d(v;C)>3|C|/4$. By Lemma 2.7(4), $d(v;I)\leqslant i/2$; by Lemma 2.8(4), $d(v;J)\leqslant j/2$; and by Lemmas 2.5(3) and 2.9, $d(v;B)\leqslant b/2$. So

$$d(v;C)\geqslant 2n/3-d(v;I\cup J\cup B)\geqslant 2n/3-(i+j+b)/2=n/2>n/2-3/4=3|C|/4.$$

Without loss of generality, assume that $i+a\leqslant b+j$. Since $i+a+j+b\geqslant n/3\geqslant 5$, $b+j\geqslant 3$. By Lemma 1.5, it suffices to show that $d(z_{p-2},z_p;D)\geqslant 3|D|/2-1$. Let $F=\{z_h\colon h \text{ is odd and } h<i\}$ and $F'=\{z_{p+1-h}\colon h \text{ is odd and } h<b+j\}$. By Lemmas 2.7(5), 2.8(5), and 2.5(3), $d(z;K)\geqslant 2$, for every $z\in F\cup F'$. Next, we show that $N_K(z_g)\cap N_K(z_h)\neq\emptyset$, for all $z_g,z_h\in F\cup F'$ with $g<h$. This is trivial, unless $d(z_g;K)=d(z_h;K)=2$. So by Lemmas 2.7(6) and 2.8(6), we are done, unless $z_g\in F$ and $z_h\in F'\cap J$. In this case, by Lemmas 2.7(6) and 2.8(6), $N_K(z_g)=K-N(z_i)$ and $N_K(z_h)=K-N(\varphi_j)$. Note that $N_K(z_i)\cap N_K(\varphi_j)\neq\emptyset$, since otherwise, $N_K(z_i)+(z_i,\ldots,\varphi_j)+N_K(\varphi_j)$ is a square cycle of length $2+1+a+m+b+1+2>2n/3$. Thus $(K-N(z_i))\cap(K-N(\varphi_j))\neq\emptyset$, and so $N_K(z_g)\cap N_K(z_h)\neq\emptyset$.

For all $z_g\in F$, $d(z_p;z_g,z_{g+1})\leqslant 1$, since otherwise, by Lemma 2.1(2), $(z_p,z_g,z_{g+1},\ldots,\mu_m)+K$ contains a square cycle of length at least $3+a+m+2>2n/3$. Thus $d(z_p;I)\leqslant i/2$. Similarly, $d(z_{p-2};I)\leqslant i/2$. For all $z_g\in F'-\{z_p\}$, $d(z_p,z_g,z_{g+1})\leqslant 1$, since otherwise, by Lemma 2.1(2) $(\mu_1,\ldots,z_{g-1},z_g,z_p)+K$ contains a square cycle of length at least $2+m+b+3>2n/3$. Thus $d(z_p;B\cup J)\leqslant(b+j)/2$. Similarly,

$d(z_{p-2}; B \cup J) \leqslant (b+j)/2 + 1$. Thus

$$\begin{aligned} d(z_{p-2}, z_p; D) &\geqslant 4n/3 - d(z_{p-2}, z_p; I \cup B \cup J) \\ &\geqslant 4n/3 - (i+b+j+1) = n-1 = 3|D|/2 - 1. \quad \square \end{aligned}$$

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